PMM U.S.S.R., Vol.51, No.4, pp.516-522, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

REPRESENTATIONS OF THE SOLUTIONS OF TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF COUPLED ELECTROELASTICITY*

V.A. SHACHNEV

The existence of special boundary value problems that are split into elementary problems characterized by one boundary condition is proved. The solutions of special boundary value problems are constructed in closed form and this enables them to be used as representations of solutions of other boundary value problems. Such representations result in especially simple systems of integral relations since some of the boundary conditions are taken into account in the representations themselves. Dielectrics with simple anisotropy, and certain classes of rhombic and monoclinic systems for which representations of solutions and integral equations are obtained for certain classical boundary value problems of electroelasticity are considered as examples.

In an anisotropic and homogeneous infinitely long cylinder with arbitrary cross-section let there exist a displacement field u and an electrical potential v that do not vary along the cylinder generator. We introduce a rectangular (x_1, x_2, x_3) coordinate system, directing the x_3 axis along the generator. We will then have $u_i = u_i(x_1, x_2)$, $i = 1, 2, 3$ and $v = v(x_1, x_2)$ for the displacement vector components and the potential.

The stress fields p_{ij} and the electric displacements q_j are determined by the following system of relations in this case /1/:

$$p_{ij} = \sum_{k=1}^{2} \Big(\sum_{l=1}^{3} c_{ijkl} \partial_k u_l + c_{kij} \partial_k v \Big) \quad (i, j = 1, 2, 3) \tag{1}$$

$$q_j = \sum_{k=1}^{2} \Big(\sum_{l=1}^{3} c_{jkl} \partial_k u_l - c_{jk} \partial_k v \Big) \quad (j = 1, 2, 3)$$

where ∂_k is the partial differentiation operator with respect to the x_k coordinate, and c_{ijkl}, c_{jkl}, c_{jk} are the elastic stiffness, piezoelectric constant, and permittivity of the material respectively. They satisfy the symmetry relationships

$$c_{jikl} = c_{ijkl}, \quad c_{klij} = c_{ijkl}, \quad c_{jlk} = c_{jkl}, \quad c_{kj} = c_{jk} \tag{2}$$

Substitution of relations (1) into the equilibrium and electrostatics equations

$$\sum_{j=1}^{2} \partial_j p_{ij} = 0 \quad (i = 1, 2, 3), \quad \sum_{j=1}^{2} \partial_j q_j = 0 \tag{3}$$

results in the fundamental system of coupled electroelasticity equations

$$\sum_{l=1}^{3} L_{il} u_l + L_{i4} v = 0, \quad i = 1, 2, 3, 4 \tag{4}$$

$$L_{il} = \sum_{j,k=1}^{2} c_{ijkl} \partial_j \partial_k, \quad L_{i4} = \sum_{j,k=1}^{2} c_{kji} \partial_k \partial_j$$

$$L_{4l} = \sum_{j,k=1}^{2} c_{jkl} \partial_j \partial_k \quad (i, l = 1, 2, 3), \quad L_{44} = -\sum_{j,k=1}^{2} c_{jk} \partial_j \partial_k$$

It follows from (2) that $L_{li} = L_{il}$ $(i, l = 1, 2, 3, 4)$.

The solutions of these equations should take given values on the boundary of their domain of definition, or forces p_i or electric charge q

$$p_i = \sum_{j=1}^{2} p_{ij} n_j \quad (i = 1, 2, 3), \quad q = -\sum_{j=1}^{2} q_j n_j \tag{5}$$

*Prikl.Matem.Mekhan., 51, 4, 662-669, 1987 should be specified on the boundary, where (n_1, n_2) is the unit vector of the normal to the boundary of the cylinder cross-section, directed along the section.

To determine the general solution of system (4) we introduce the resolving function χ_s by means of the formulas

$$u_l = \sum_{s=1}^{4} M_{ls}\chi_s, \quad v = \sum_{s=1}^{4} M_{4s}\chi_s \tag{6}$$
$$\sum_{l=1}^{4} L_{il}M_{ls} = \delta_{is}L, \quad L = \det \| L_{ij} \|$$

where $M_{ls} = M_{sl}$ are cofactors of the matrix $\| L_{ij} \|$, and δ_{is} is the Kronecker delta.

Substituting relations (6) into system (4) we obtain that all χ_s should satisfy the equation

$$L\chi = 0, \quad L = \prod_{n=1}^{8} (a_n\partial_1 + b_n\partial_2) \tag{7}$$

where L is an eighth-order homogeneous differential operator with constant coefficients and partial derivatives in two variables x_1 and x_2. Such an operator can always be expanded in linear factors as shown above. The sum of the solutions of an equation of the form

$$(a\partial_1 + b\partial_2)\, \chi = 0 \tag{8}$$

will be the kernel of this operator.

The solution of each such equation has the form

$$\chi = \operatorname{Re} f\,(x_1 + kx_2) \tag{9}$$

where f is an arbitrary analytic function of $z = x_1 + kx_2$.

To determine k, we substitute (9) directly into (7) and we then obtain an equation for k which we call characteristic

$$l\,(k) = 0 \tag{10}$$

where the polynomial l is obtained from L by replacing (∂_1, ∂_2) by $(1, k)$.

We will consider here just the case of pairwise distinct roots of the characteristic equation. Since this equation has real coefficients and corresponds to an even-order elliptic operator, then four pairs of conjugate complex numbers $k_n = \alpha_n \pm i\beta_n$, $\beta_n > 0$ will be its roots. Therefore, the general solution of (7) can be represented in the form

$$\chi = \operatorname{Re} \sum_{n=1}^{4} f_n(z_n), \quad z_n = x_1 + k_nx_2, \quad k_n = \alpha_n + i\beta_n \tag{11}$$

where only roots with positive imaginary part are taken, and f_n are arbitrary analytic functions. We will consequently have the following representation of the solutions (6)

$$u_l = \operatorname{Re} \sum_{s,\, n=1}^{4} m_{ls}(k_n) f_{ns}^{(8)}(z_n), \quad v = \operatorname{Re} \sum_{s,\, n=1}^{4} m_{4s}(k_n) f_{ns}^{(8)}(z_n) \tag{12}$$

where the polynomials $m_{ls}(k)$ are obtained from M_{ls} by replacing (∂_1, ∂_2) by $(1, k)$, and f_{ns} are arbitrary analytic functions. Their number is not so large, as is indicated in (12), since $m_{ls}(k_n)$ will not be linearly independent. Consequently, to obtain the final form of the solutions (12), we proceed as follows. We consider the preceding reasoning to be the proof that the general solution of system (4) can be sought at once in the form

$$u_l = \operatorname{Re} \sum_{n=1}^{4} m_{ln}g_n(z_n), \quad v = \operatorname{Re} \sum_{n=1}^{4} m_{4n}g_n(z_n) \tag{13}$$

To determine $m_{ln} = m_l(k_n)$ we do not use system (4) but Eqs. (3) by first determining p_{ij} and q_j from (1):

$$p_{ij} = \operatorname{Re} \sum_{n=1}^{4} \Big(\sum_{l=1}^{3} (c_{ij1l} + c_{ij2l}k_n)\, m_{ln} + (c_{1ij} + c_{2ij}k_n)\, m_{4n} \Big)\, g_n' \tag{14}$$
$$q_j = \operatorname{Re} \sum_{n=1}^{4} \Big(\sum_{l=1}^{3} (c_{j1l} + c_{j2l}k_n)\, m_{ln} - (c_{1j} + c_{2j}k_n)\, m_{4n} \Big)\, g_n'$$

We introduce the differential forms

$$p_i ds = p_{i1}dx_2 - p_{i2}dx_1, \quad qds = -q_1dx_2 + q_2dx_1$$

obtained from (5) taking $(n_1, n_2)\, ds = (dx_2, -dx_1)$ into account, where ds is the differential of the length of the contour of the transverse section domain. Calculating the external differential of these forms, we will have, by virtue of (3),

$$\partial (p_{i1}dx_2 - p_{i2}dx_1) = (\partial_1 p_{i1} + \partial_2 p_{i2})\, dx_1 \wedge dx_2 = 0$$
$$\partial (-q_1 dx_2 + q_2 dx_1) = -(\partial_1 q_1 + \partial_2 q_2)\, dx_1 \wedge dx_2 = 0$$

By virtue of Poincaré's theorem it hence follows that there exist (at least locally) such null-forms P_i and Q that

$$p_i ds = dP_i \quad (i = 1, 2, 3), \quad q ds = dQ \tag{15}$$

and moreover

$$p_{i1} = \partial_2 P_i, \quad p_{i2} = -\partial_1 P_i, \quad q_1 = -\partial_2 Q, \quad q_2 = \partial_1 Q \tag{16}$$

Now we represent the functions P_i and Q in the same form as u_l and v:

$$P_i = \operatorname{Re} \sum_{n=1}^{4} \sigma_{in} g_n (z_n), \quad Q = \operatorname{Re} \sum_{n=1}^{4} \sigma_{4n} g_n (z_n) \tag{17}$$

where according to (15)

$$P_i = \int_{s_0}^{s} p_i \, ds + c_i, \quad Q = \int_{s_0}^{s} q \, ds + c$$

are determined on the section boundary, apart from constants, and are the integral forces and integral charge.

Subsituting (17) into (16) we obtain

$$\begin{aligned} p_{i1} &= \operatorname{Re} \sum_{n=1}^{4} \sigma_{in} k_n g_n{}', \quad p_{i2} = -\operatorname{Re} \sum_{n=1}^{4} \sigma_{in} g_n{}' \\ q_1 &= -\operatorname{Re} \sum_{n=1}^{4} \sigma_{4n} k_n g_n{}', \quad q_2 = \operatorname{Re} \sum_{n=1}^{4} \sigma_{4n} g_n{}' \end{aligned} \tag{18}$$

Comparing (18) with (14) we obtain (omitting the subscript n) the expressions

$$\begin{aligned} \sigma_i &= -\sum_{l=1}^{3} (c_{i21l} + c_{i22l} k)\, m_l - (c_{21i} + c_{22i} k)\, m_4 \\ \sigma_4 &= \sum_{l=1}^{4} (c_{21l} + c_{22l} k)\, m_l - (c_{21} + c_{22} k)\, m_4 \end{aligned} \tag{19}$$

as well as the additional relationships

$$\begin{aligned} &\sum_{l=1}^{3} (c_{i11l} + c_{i12l} k)\, m_l + (c_{11i} + c_{12i} k)\, m_4 = k\sigma_i \\ &-\sum_{l=1}^{3} (c_{11l} + c_{12l} k)\, m_l + (c_{11} + c_{12} k)\, m_4 = k\sigma_4 \end{aligned} \tag{20}$$

Eliminating σ_i from (19) and (20), we arrive at a system of equations to determine k and m_i

$$\begin{aligned} &\sum_{l=1}^{3} (c_{i11l} + (c_{i12l} + c_{i21l})\, k + c_{i22l} k^2)\, m_l + \\ &\quad (c_{11i} + (c_{12i} + c_{22i})\, k + c_{22i} k^2)\, m_4 = 0 \\ &-\sum_{l=1}^{3} (c_{11l} + (c_{12l} + c_{21l})\, k + c_{22l} k^2)\, m_l + (c_{11} + 2c_{12} k + c_{22} k^2)\, m_4 = 0 \end{aligned} \tag{21}$$

The determinant of this system is the characteristic polynomial (10), and its vanishing results in an equation for determining k. Subsequently, $m_{in} = m_i\,(k_n)$ are determined from system (21).

Going over to the boundary value problems, we note that by virtue of (15) it is formally indistinguishable as to what is given on the boundary, p_i and q or P_i and Q. It is more convenient to specify P_i and Q, since the nature of the boundary conditions for the desired functions is the same in this case as when giving u_i and v, and this enables to formulate the following problem: formulate boundary conditions in the form of linear combinations of u_i, P_i, v, Q such that the solution of the boundary value problem reduces to a sequential solution of the boundary value problems for just one of the functions g_n each time. In this case we will say that the boundary value problem is solvable. If the boundary value problem allows a separate solution of the boundary value problems for each function g_n, we will say that such a problem is absolutely solvable. We still note that each of the functions g_n has its domain of definition that is obtained from the domain of the cylinder cross-section by the transformation

$$z_n = A_n z, \quad A_n = \left\| \begin{matrix} 1 & \alpha_n \\ 0 & \beta_n \end{matrix} \right\|, \quad z = \left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\| \tag{22}$$

Theorem. If the roots of the characteristic Eq.(10) are pairwise distinct, then absolutely solvable boundary value problems exist.

Proof. The systems of Eqs.(19) and (20) can be written in the form of a single matrix equation

$$(A + kB) W, \quad W = \mathrm{col}\,(m_1, \ldots, m_4, \sigma_1, \ldots, \sigma_4) \tag{23}$$

$$A = \left\| \begin{matrix} A_{11} & E \\ A_{21} & 0 \end{matrix} \right\|, \quad B = \left\| \begin{matrix} B_{11} & 0 \\ B_{21} & E \end{matrix} \right\|$$

$$A_{21} = \left\| \begin{matrix} c_{1111} & c_{1112} & c_{1113} & c_{111} \\ c_{2111} & c_{2112} & c_{2113} & c_{112} \\ c_{3111} & c_{3112} & c_{3113} & c_{113} \\ -c_{111} & -c_{112} & -c_{113} & c_{11} \end{matrix} \right\|, \quad B_{11} = \left\| \begin{matrix} c_{1221} & c_{1222} & c_{1223} & c_{221} \\ c_{2221} & c_{2222} & c_{2223} & c_{222} \\ c_{3221} & c_{3222} & c_{3223} & c_{223} \\ -c_{221} & -c_{222} & -c_{223} & c_{22} \end{matrix} \right\|$$

where E is the unit matrix. Only the matrices A_{21} and B_{11} are of interest.

We will show that the determinants of the matrices A and B are different from zero, where it is sufficient to show this for just one of them by virtue of relationships (23). As is well-known /1/, the internal energy for dielectrics has the form

$$\mathrm{E} = \frac{1}{2} \sum_{i,j=1}^{3} p_{ij} \varepsilon_{ij} + \frac{1}{2} \sum_{j=1}^{3} q_j e_j$$

where $2\varepsilon_{ij} = \partial_i u_j + \partial_j u_i$, $e_j = -\partial_j v$ are the components of the strain tensor and electric intensity vector. Since the internal energy is a positive-definite quadratic form for arbitrary ε_{ij} and e_j, then it also remains such in the case when $\varepsilon_{11} = 0$, $\varepsilon_{31} = 0$, $e_1 = 0$ ($\varepsilon_{33} = 0$, $e_3 = 0$, since the problem is two-dimensional). But then this form can be written as

$$\mathrm{E} = \frac{1}{2} V^T B_{11} V > 0, \quad V = \mathrm{col}\,(2\varepsilon_{12}, \varepsilon_{22}, 2\varepsilon_{23}, e_2) \neq 0$$

It is now obvious that $\det B = \det B_{11} > 0$ and it follows from (23) that k is an eigenvalue of the matrix $-B^{-1}A$ and satisfies the equation $\det(A + kB) = 0$ which is the characteristic Eq.(10).

Since the eigenvalues k_n are pairwise distinct by the condition of the theorem, then the natural columns

$$W_n = \mathrm{col}\,(m_{1n}, \ldots m_{4n}, \sigma_{1n}, \ldots \sigma_{4n}) \quad (n = 1, 2, 3, 4)$$

and their conjugates will be linearly independent.

We now represent the expressions for u_i, P_i, v, Q in the form

$$\sum_{n=1}^{4} (m_{in} g_n + \overline{m_{in} g_n}) = 2u_i, \quad \sum_{n=1}^{4} (\sigma_{in} g_n + \overline{\sigma_{in} g_n}) = 2P_i$$

$$\sum_{n=1}^{4} (m_{4n} g_n + \overline{m_{4n} g_n}) = 2v, \quad \sum_{n=1}^{4} (\sigma_{4n} g_n + \overline{\sigma_{4n} g_n}) = 2Q$$

and we consider them as a system of equations in g_n and $\bar{g}_n$. The determinant of this system differs from zero since it is the determinant of a matrix of linearly independent columns $W_n, \overline{W}_n$ $(n = 1, 2, 3, 4)$ and, therefore, the system has a solution of g_n and $\bar{g}_n$, or equivalently, for $\mathrm{Re}\, g_n$ and $\mathrm{Im}\, g_n$.

We consider these solutions on the boundaries of the corresponding domains. Since the boundary values of the real and imaginary parts of a function analytic in the domain cannot be arbitrary, we form linear combinations of them which we write in the form

$$\theta_n \operatorname{Re} g_n + \omega_n \operatorname{Im} g_n = \sum_{i=1}^{3} (a_{ni} u_i + b_{ni} P_i) + a_{n4} v + b_{n4} Q$$

where θ_n, ω_n are arbitrary piecewise-smooth functions of points of the domain boundary. In sum, we arrive at four individual boundary value problems of the theory of functions of a complex variable: to reproduce a function analytic in a domain by the boundary values of a linear combination of its real and imaginary parts. The theorem is proved.

It is now clear from the proof of the theorem how solvable boundary value problems of two-dimensional electroelasticity must be formulated. However, it is possible to proceed differently by relying on the assertion of the theorem. We still form linear combinations with determined coefficients.

$$\Psi_i = \sum_{j=1}^{3} (a_{ij} u_j + b_{ij} P_j) + a_{i4} v + b_{i4} Q$$

Then according to (13) and (17) we will have the following relationships on the boundary

$$\operatorname{Re} \sum_{n=1}^{4} w_{in} g_n = \Psi_i, \quad w_{in} = \sum_{j=1}^{4} (a_{ij} m_{jn} + b_{ij} \sigma_{jn})$$

Now the matrices $\| a_{ij} \|$ and $\| b_{ij} \|$ should be selected such that the matrix $\| w_{in} \|$ will become diagonal, - this for the case of absolute stability. In the case of simple solvability it is sufficient to reduce the matrix $\| w_{in} \|$ to triangular form. Then much arbitrariness remains in the matrices $\| a_{ij} \|$ and $\| b_{ij} \|$, and this broadens the class of solvable boundary value problems; in particular, solvability of isotropic problems is possible /2/.

It will be seen from the examples presented below that not every boundary value problem is solvable. In particular, the problem in which all the displacement components and the electric potential are given is not solvable. However, the solutions obtained can be used as representations of solutions of any other boundary value problems, for instance, just as is done in /3/. We consequently arrive at a system of integral equations on the boundary of the domain which are convenient in the sense that they contain not all the boundary functions as unknowns but only some of them, since the rest, at least two boundary conditions, has already been taken into account in the representation of the solution.

Example 1. We consider dielectrics of a class 2mm rhombic system (the x_1 axis is a double axis of symmetry, two coordinate planes containing the x_1 axis, is a plane of symmetry). In this case only $c_{iijj}, c_{ijij}, c_{ij}, c_{111}, c_{122}, c_{212}, c_{222}, c_{133}, c_{331}, c_{323}$, can be constants different from zero, where the last three do not participate in the two-dimensional problem. System (21) takes the form

$$\begin{aligned}
&(c_{1111} + c_{1212} k^2) m_1 + (c_{1122} + c_{1212}) k m_2 + (c_{111} + c_{212} k^2) m_4 = 0 \\
&(c_{1122} + c_{1212}) k m_1 + (c_{1212} + c_{2222} k^2) m_2 + (c_{212} + c_{122}) k m_4 = 0 \\
&c_{3131} + c_{2323} k^2 = 0 \\
&(c_{111} + c_{212} k^2) m_1 + (c_{212} + c_{122}) k m_2 - (c_{11} + c_{22} k^2) m_4 = 0
\end{aligned} \tag{24}$$

The system decouples and $k_3, \bar{k}_3$ are determined from the third equation. The determinant of the remaining system has the form $l = l_6 k^6 + l_4 k^4 + l_2 k^2 + l_0$ and k_1, k_2, k_4 and their conjugates are determined from the equation $l = 0$.

The form of the equation enables us to extract the case when all the roots of the equation $l = 0$ are pure imaginary, just as the third equation in (24). In this case we have two separate systems of boundary conditions

$$\sum_n{}' (m_{1n}, m_{4n}, \sigma_{2n}) \operatorname{Re} g_n = (u_1, v, P_2)$$

$$\sum_n{}' (i m_{2n}, i\sigma_{1n}, i\sigma_{4n}) \operatorname{Im} g_n = (u_2, P_1, Q)$$

(the prime denotes that the number $n = 3$ is omitted). It is possible to determine

$$\operatorname{Re} g_n = \alpha_{n1} u_1 + \alpha_{n2} P_2 + \alpha_{n4} v \tag{25}$$

$$\operatorname{Im} g_n = \beta_{n1} P_1 + \beta_{n2} u_2 + \beta_{n4} Q \tag{26}$$

respectively, from the first and second system.

There is an individual boundary condition for u_3 and P_3, and consequently, it will not be taken into account henceforth.

Separate boundary value problems are consequently obtained in which it is necessary to restore the analytic function in the domain by means of its given real or imaginary part on the boundary. We will represent the solutions of these problems in closed form.

Let the function $z_n = f_n(\zeta_n)$ map some standard domain of the complex plane, the upper half-plane, say, into the domain of definition of the function g_n. Then the solution of the boundary value problem can be written by means of the Schwartz formula in the form

$$g_n(z_n) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\alpha_{n1} u_1 + \alpha_{n2} P_2 + \alpha_{n4} v}{\xi_n - \zeta_n(z_n)} \, d\xi_n + i c_n \tag{27}$$

or

$$g_n(z_n) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta_{n1} P_1 + \beta_{n2} u_2 + \beta_{n4} Q}{\xi_n - \zeta_n(z_n)} \, d\xi_n + c_n \tag{28}$$

where u_i, P_i, v, Q are given as functions of the coordinate $\xi_n = f_n^{-1}(z_n)$, and $(z_n) = x_{n1} + i x_{n2}$ belongs to the boundary of the domain of definition of g_n.

Substituting (27) and (28) into (13), we obtain representations of the solutions of the boundary value problems in this case for dielectrics of the given class.

Let us examine the boundary value problem in which displacement components and the electric potential are given. If (27) is selected as the representation of the solution, then it remains just to consider the boundary condition for u_2. From (13) we have the representation $(u_4 = v)$

$$u_j = \mathrm{Re} \sum_{n=1}^{4} \frac{m_{jn}}{\pi i} \int_{-\infty}^{\infty} \frac{\alpha_{n1} u_1 + \alpha_{n2} P_2 + \alpha_{n4} v}{\xi_n - \zeta_n(z_n)} \, d\xi_n + c_j \quad (j = 1, 2, 4) \tag{29}$$

Transferring z_n on the boundary of the domain into the equation for $j = 2$, according to the Sokhotskii-Plemelj formulas, we obtain an integral equation for the unknown boundary function in this case, the integral force P_2

$$\mathrm{Re} \sum_{n=1}^{4}{}' m_{2n} \alpha_{n2} \left[P_2(\xi) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{P_2(\tau(\tau_n))}{\tau_n - \xi_n(\xi)} \, d\tau_n \right] = R(\xi) \tag{30}$$

$$R(\xi) = u_2(\xi) - \mathrm{Re} \sum_{n=1}^{4}{}' m_{2n} \left[\alpha_{n1} u_1(\xi) + \alpha_{n4} v(\xi) + \right.$$

$$\left. \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\alpha_{n1} u_1(\tau(\tau_n)) + \alpha_{n4} v(\tau(\tau_n))}{\tau_n - \xi_n(\xi)} \, d\tau_n \right] - c_2$$

The dependence $\xi_n = \xi_n(\xi)$ is determined from the relationship $f_n(\xi_n) = A_n f(\xi)$ where the function $z = f(\zeta)$ maps the deformed domain onto the upper half-plane $\eta \geqslant 0$, $\zeta = \xi + i\eta$.

Example 2. In the class 002 of a monoclinic system (the x_3 axis is the axis of double symmetry(the constraints c_{ii23}, c_{ii31}, c_{1223}, c_{1231}, c_{1jj}, c_{122}, c_{2jj}, c_{212}, c_{323}, c_{331}, c_{23}, c_{31} equal zero, and system (21) splits into two independent systems

$$\begin{aligned} &(c_{1111} + 2c_{1112}k + c_{1212}k^2)m_1 + (c_{1112} + (c_{1122} + c_{1212})k + c_{1222}k^2)m_2 = 0 \\ &(c_{1112} + (c_{1122} + c_{1212})k + c_{1212}k^2)m_1 + (c_{1212} + 2c_{1222}k + c_{2222}k^2)m_2 = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} &(c_{3131} + 2c_{3123}k + c_{2323}k^2)m_3 + (c_{113} + (c_{123} + c_{213})k + c_{223}k^2)m_4 = 0 \\ &(c_{113} + (c_{123} + c_{213}k) + c_{223}k^2)m_3 - (c_{11} + 2c_{12}k + c_{22}k^2)m_4 = 0 \end{aligned} \tag{32}$$

Hence it follows from (19) that

$$\begin{aligned} \sigma_1 &= -(c_{1211} + c_{1221}k)m_1 - (c_{1212} + c_{1222}k)m_2 \\ \sigma_2 &= -(c_{2211} + c_{2221}k)m_1 - (c_{2212} + c_{2222}k)m_2 \end{aligned} \tag{33}$$

$$\begin{aligned} \sigma_3 &= -(c_{3213} + c_{3223}k)m_3 - (c_{123} + c_{223}k)m_4 \\ \sigma_4 &= (c_{213} + c_{223}k)m_3 - (c_{21} + c_{22}k)m_4 \end{aligned} \tag{34}$$

Equating the determinants of systems (31) and (32) to zero we obtain two characteristic equations whose roots are $k_1, k_2, \bar{k}_1, \bar{k}_2$ and $k_3, k_4, \bar{k}_3, \bar{k}_4$, respectively.

It is seen that the problem splits into two; one is described in the boundary value terminology u_1, u_2, P_1, P_2 and the other in the terminology u_3, P_3, v, Q. A special case of the first problem $(c_{1112} = c_{1222} = 0)$ was considered in /3, 4/. We examine here only the second problem, where for simplicity we set c_{1112}, c_{1222}, c_{2231}, c_{113}, c_{123}, c_{223}, c_{12} equal to zero, corresponding to a dielectric of class 422. In this case the characteristic equation has only pure imaginary roots.

Determining m_i and σ_i from (32) and (34), we obtain two systems of boundary conditions for g_n: one is expressed in terms of u_3 and Q and the other in terms of P_3 and v. We consider only the first. In this case the solution has the form

$$\operatorname{Re} g_n = (-1)^n [u_3 + c_{11}^{-1} c_{213}^{-1} (c_{11} + c_{22} k_{7-n}^2) Q]/\Delta$$
$$\Delta = c_{22} (k_4^2 - k_3^2)$$

and, as before, g_n is restored in the domain.

In order to examine other boundary value problems, we present a representation of the remaining functions. From (13) and (17) we obtain

$$\left\| \begin{matrix} P_3 \\ v \end{matrix} \right\| = \operatorname{Re} \sum_{n=3}^{4} \frac{(-1)^n}{\pi i} \left\| \begin{matrix} \sigma_{3n} \\ m_{4n} \end{matrix} \right\| \int_{-\infty}^{\infty} \frac{u_3 + c_{11}^{-1} c_{213}^{-1} (c_{11} + c_{22} k_{7-n}^2) Q}{\Delta (\xi_n - \zeta_n (z_n))} d\xi_n + C$$

Now, if the displacement and electrical potential v are given on the boundary, then according to the Sokhotskii-Plemelj formulas we obtain the following integral equation for Q:

$$\sum_{n=3}^{4} \frac{(c_{11} + c_{22} k_{7-n}^2) \operatorname{Im} m_{4n}}{c_{11} c_{213} \Delta \pi} \int_{-\infty}^{\infty} \frac{Q(\tau(\tau_n))}{\tau_n - \xi_n(\xi)} d\tau_n = R(\xi) \tag{35}$$

$$R(\xi) = v(\xi) - \sum_{n=1}^{4} \frac{\operatorname{Im} m_{4n}}{\Delta \pi} \int_{-\infty}^{\infty} \frac{u_3(\tau(\tau_n))}{\tau_n - \xi_n(\xi)} d\tau_n - c_4$$

We consider a specific example for the second problem. Let the domain of the cylindrical body section be the exterior of a parabola $y \geqslant 4a^2 (x + a^2)$, and a the parameter of the parabola $(x_1 = x, x_2 = y)$ and the displacement u_3 and the potential v given on the domain boundary, where $u_3(\infty) = v(\infty) = 0$.

The roots of the characteristic equation of the system (32) have the form $k_n = i\beta_n, \beta_n > 0$ $(n = 3, 4)$ and $m_{3n} = c_{11} - c_{22}\beta_n^2$, $m_{4n} = ic_{213}\beta_n$ is a solution of this system.

The transformations $z_n = A_n z$ $(n = 3, 4)$ or $x_n = x, y_n = \beta_n y$ map this domain into the domain $y_n^2 \geqslant 4\beta_n^2 a^2 (x_n + a^2)$.

The functions mapping the half-plane conformally into the transformed domains, will have the form

$$z_n = \beta_n^2 (\zeta_n^2 + i2a\zeta_n) - a^2, \quad \zeta_n = \xi_n + i\eta_n$$

and in particular, for $\beta_n = 1$, we obtain a mapping of the half-plane into a given section domain such that by virtue of the relationship $z_n = A_n z$ we will have $\xi_n = \beta_n^{-1}\xi$ on the section boundary. This last circumstance makes the kernel of the integrals in (35) similar $(\tau_n - \xi_n(\xi))^{-1} = \beta_n (\tau - \xi)^{-1}$, which enables us to solve (35) by inverting the Cauchy operator. We consequently find

$$Q(\xi) = -c_{213} u_3(\xi) + \frac{c_{22}(\beta_3 + \beta_4)}{\pi} \int_{-\infty}^{\infty} \frac{v(\tau)\, d\tau}{\tau - \xi} + c$$

and the problem is solved.

REFERENCES

1. NOWACKI W., Electromagnetic Effects in Deformable Solids, PAN, Warsaw, 1983.
2. SHACHNEV V.A., On the solvability of the plane problem of elasticity theory for an isotropic domain, Nauch. Tr. Mosk. Lesotekh. Ins., 140, 1982.
3. SHACHNEV V.A., Some representations of the solutions of boundary value problems of two-dimensional static thermoelasticity, PMM, 45, 1, 1981.
4. SHACHNEV V.A., On the state of stress of an orthotropically elastic plane domain in the neighbourhood of an angular point, PMM, 46, 1, 1982.

Translated by M.D.F.